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An extension of the Vu–Sine theorem and compact-supercyclicity

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Abstract

If $(T_t)_{t\geqslant 0}$ is a bounded C_0-semigroup in a Banach space X and there exists a compact subset $K\subseteq X$ such that

$$\liminf_{t\to\infty}\rho(T_t x, K)=0 \quad \bigl(\forall x\in X,\ \|x\|\leqslant 1\bigr),$$

then there exists a finite-dimensional subspace $L\subseteq X$ such that

$$\lim_{t\to\infty}\rho(T_t x, L)=0 \quad (\forall x\in X).$$

If $T: X\to X$ (X is real or complex) is supercyclic and $(\|T^n\|)_n$ is bounded then $(T^n x)_n$ vanishes for every $x\in X$.

We define the "compact-supercyclicity." If $\dim X=\infty$ then X has no compact-supercyclic isometries.

Keywords: C_0-semigroup; Power bounded operator; Supercyclicity; Almost-periodic representation

0. Introduction

Let X be a Banach space, by B_X we denote the unit ball in X. For a subset $Y\subseteq X$ and $x\in X$ we denote by $\rho(x,Y)$ the distance between x and Y.

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Definition 1. Let $T : X \to X$ be a map. A subset $K \subseteq X$ is called an *attractor* for T if

$$\forall x \in B_X \quad \lim_{n\to\infty} \rho\bigl(T^n x, K\bigr) = 0. \tag{1}$$

The definition of an attractor for a C_0-semigroup is similar. It is known that for a linear power bounded operator (and for a bounded C_0-semigroup) the existence of a compact attractor implies the existence of an invariant finite-dimensional subspace $L \subseteq X$ and an invariant subspace $X_0 \subseteq X$ such that $X = X_0 \oplus L$ and the semigroup $(T^n)_n$ is isomorphic to the direct product of semigroups

$$T^n = (T|_{X_0})^n \oplus (T|_L)^n : X_0 \oplus L \to X_0 \oplus L, \quad \forall x_0 \in X_0,\ T^n x_0 \to 0. \tag{2}$$

This theorem was proved in [4] for the Markov semigroups in L_1. Its general case was proved by Vu [10] and Sine [11]. We call this result the Vu–Sine theorem.

It turns out that the conclusion of the Vu–Sine theorem remains true if there exists only "occasionally attracting" compact set K:

$$\liminf_{t\to\infty} \rho(T_t x, K) = 0 \quad (\forall x \in B_X). \tag{1$'$}$$

The papers [10,11] use the results of Jacobs [6] and de Leeuw and Glicksberg [7] on spectral decomposition of weakly almost periodic semigroup. We base on a more elementary fact, the non-emptiness of the essential spectrum.

In the first part of the paper we prove the following theorem:

Theorem 1. *Let* $\dim X = \infty$. *For any isometry* $T : X \to X$ *there are no occasionally attracting compact sets.*

The second part of the paper is devoted to the application of Theorem 1 to the above-mentioned strengthening of the results of [4,10,11].

Definition 2. Let $x \in X$ or $x \subseteq X$. Denote by $O(x) = \bigcup_{n=0}^{\infty} T^n x$ the orbit of x.

Definition 3. A vector a is called a *recurrent vector* if $\liminf_{n\to\infty} \|T^n a - a\| = 0$.

It is easy to see that if T is power bounded then $a \in X$ is a recurrent vector if and only if a is a limit point of the orbit of some $x \in X$.

Lemma 1. *Let* $T : X \to X$ *and* $\|T\| \leqslant 1$. *If a is a recurrent vector, then the subspace* $L(a) = \mathrm{cl}(\mathrm{span}(O(a)))$ *consists of recurrent vectors and* $T : L(a) \to L(a)$ *is an isometry.*

Then we prove the generalizations of the Vu–Sine theorem from Theorem 1:

Theorem 2. *Let $T : X \to X$ be a power bounded operator. If there exists a compact set K such that* (1$'$) *holds, then the semigroup* $(T^n)_{n=0}^{\infty}$ *is "asymptotically finite-dimensional," i.e. there exists an invariant subspace $L \subseteq X$,* $\dim(L) < \infty$ *such that, for every $x \in X$,* $\lim_{n\to\infty} \rho(T^n x, L) = 0$ *and decomposition* (2) *holds. The space L is generated by all recurrent vectors of T.*

Theorem 3. *Let $(T_t)_{t \geqslant 0}$ be a bounded C_0-semigroup in a Banach space X. If there exists an occasionally attracting compact set $K \subseteq X$, then the semigroup T is asymptotically finite-dimensional.*

The last part of the paper is devoted to another application of Theorem 1.

Let X be a real or complex infinite-dimensional Banach spaces and $F \in \{\mathbb{R}, \mathbb{C}\}$. An operator $T : X \to X$ is called *supercyclic* if there exists a vector $k \in X$ such that the set $F \cdot O(k)$ is dense in X. The corresponding vector k is called *supercyclic*.

The following results were proved for complex X in [1,8]:

Theorem 4. *If $T : X \to X$ is isometry, then T is not supercyclic. Moreover, if T is power bounded and supercyclic, then $T^n x$ vanishes for every $x \in X$.*

Both [1] and [8] make use of the Godement theorem [5]: every isometry of complex X has an invariant proper closed subspace.

We deduce Theorem 4 (in the real and complex cases) from Theorem 1. The proof is based on the following lemma:

Lemma 4. *Let $\|T\| \leqslant 1$. If $T^n a \nrightarrow 0$ and there exist λ_k and n_k such that $\lambda_k T^{n_k} a \to a$ (in particular, if a is supercyclic), then a is a recurrent vector.*

Remark. In [8] Miller proved that an isometry of a complex X cannot even be *finite-supercyclic*, i.e., for any finite set $K \subseteq Z$, the set $F \cdot O(K)$ is not everywhere dense in Z. But after that, Peris [9] showed that, for locally convex spaces, finite-supercyclicity is equivalent to supercyclicity. A weaker property is *N-supercyclicity* [2,3]. An operator T is N-supercyclic if there exists a finite-dimensional subspace $L \subset X$ such that $X = Cl(\mathbb{C} \cdot O(B_L))$, or, equivalently, $B_X \subset Cl(O(L))$. Following this tradition, we may call $T : X \to X$ *compact-supercyclic* if there is a compact set $K \subseteq X$ such that $B_X \subset Cl(O(K))$. This definition is contensive if $\dim X = \infty$. Notice: the condition "$\exists K\colon\ X = Cl(\mathbb{C} \cdot O(K))$" holds even for identity of separable X!

For example, if $T : X \to X$ is an isometry then the presence of an occasionally attracting compact set for T is equivalent to compact-supercyclicity of T^{-1} (cf. the proof of Theorem 4). Therefore, we can reformulate Theorem 1 as follows:

If $\dim X = \infty$ *then X has no compact-supercyclic isometries.*

1. Proof of Theorem 1

First we consider the case of a complex X. Let $\sigma_{\text{ess}}(T)$ be the essential spectrum. If $\lambda \in \sigma_{\text{ess}}(T)$ then $\dim\ker(T - \lambda) = \infty$ or the $\operatorname{Im}(T - \lambda)$ is not closed in X.

We say that a bounded sequence $z_n \in X$ is *sparse* if it contains no converging subsequence. Let us show that it is possible to assign to each $\lambda \in \sigma_{\text{ess}}(T)$ a sparse sequence of "approximate eigenvectors" $z_n \in B_X$, i.e. such that $Tz_n - \lambda z_n \to 0$. Borrowing the terminology from the theory of self-adjoint operators in Hilbert spaces, we call such a sequence z_n a *Weyl sequence*.

Lemma 2. *For each $\lambda \in \sigma_{\text{ess}}(T)$, there exists a Weyl sequence z_n.*

Proof. Put $S = (T - \lambda) : X \to X$. We have: either $\dim\ker S = \infty$ or $S(X)$ is not closed in X. If $\dim\ker S = \infty$, then the statement is obvious.

If $\dim\ker S < \infty$ then $\ker S$ has a closed complement $V \subseteq X$. Consider the operator $S|_V : V \to X$. The kernel of $S|_V$ is zero and its image $S|_V(V) = S(X)$ is not closed in X. Therefore the inverse operator $(S|_V)^{-1} : S(X) \to V$ is unbounded and there exists a sequence $z_n \in V$,

$\|z_n\| = 1$ such that $Sz_n \to 0$. The sequence z_n has no limit points, since they would be nonzero elements of the kernel of $S|_V$. □

If $z_n \in X$ is a sparse sequence and $Tz_n - \lambda z_n \to 0$ then

$$\forall k \in \mathbb{N} \quad \left\|T^k z_n - \lambda T^{k-1} z_n\right\| = \|Tz_n - \lambda z_n\| \to 0. \tag{3}$$

Suppose that K is an occasionally attracting compact set. For each $n \in \mathbb{N}$, there exist a number k_n and $a_n \in K$ such that $\|T^{k_n} z_n - a_n\| < \frac{1}{n}$. Switching to a subsequence, one can assume that $a_n \to a$ and $\|T^{k_n} z_n - a\| \to 0$, i.e. $T^{k_n} z_n \to a$. It follows from (3) that $Ta = \lambda a$. In particular, the $\mathbb{Z}$-orbit $\{T^n a \mid n \in \mathbb{Z}\}$ of a lies in some one-dimensional subspace $L(a) \subseteq X$. But

$$\rho\big(z_n, L(a)\big) \leqslant \left\|z_n - T^{-k_n} a\right\| = \left\|T^{k_n} z_n - a\right\| \to 0,$$

i.e., the sequence z_n approaches a one-dimensional subspace and thus cannot be sparse. The theorem is proved in the complex case.

The real case requires the following auxiliary lemma.

Lemma 3 *(An analog of spectrum in real space). Let X be a real space and let $T: X \to X$ be a bounded operator. There exist two numbers $r, s \in \mathbb{R}$ such that the operator $S := T^2 + rT + s$ is not bijective. Moreover, if* $\dim X = \infty$, *then there exist $r, s \in \mathbb{R}$ and a sparse sequence x_n such that $T^2 x_n + rTx_n + sx_n \to 0$.*

Proof. Any complex λ is a root of the real polynomial

$$S_\lambda(t) = (t - \lambda)(t - \bar{\lambda}) = t^2 - t(\lambda + \bar{\lambda}) + |\lambda|^2.$$

Consider the complexification: $T_{\mathbb{C}}: X_{\mathbb{C}} \to X_{\mathbb{C}}$, $T_{\mathbb{C}}(x + iy) = Tx + iTy$. If $\lambda \in \sigma(T_{\mathbb{C}})$, then the operator $T_{\mathbb{C}} - \lambda$ is not bijective, hence the operator $S_\lambda(T_{\mathbb{C}})$ is not bijective either. On the other hand, the coefficients of the polynomial S_λ are real, so $S_\lambda(T_{\mathbb{C}}) = (S_\lambda(T))_{\mathbb{C}}$; therefore $S_\lambda(T): X \to X$ is not bijective as well.

If $\dim X = \infty$, let $\lambda \in \sigma_{\mathrm{ess}}(T_{\mathbb{C}})$ and $z_n = x_n + iy_n \in X_{\mathbb{C}}$ be the corresponding Weyl sequence. Then $S_\lambda(T_{\mathbb{C}})z_n \to 0$. But then $S_\lambda(T)x_n \to 0$ and $S_\lambda(T)y_n \to 0$. The sequences x_n and y_n do not have to be sparse, but if in the sequence $y_n \in X$ of the imaginary parts of $z_n \in X_{\mathbb{C}}$ there can be found a converging subsequence $y_{n_k} \in X$ then the corresponding subsequence of the real parts $x_{n_k} \in X$ is certainly sparse. The lemma is proved. □

Example. If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation on $\alpha \in (0, \pi)$, then $T^2 - \frac{\sin 2\alpha}{\sin \alpha} T + 1 = 0$.

Now we finish the proof of Theorem 1 in the real case. Let x_n be a sparse sequence such that $T^2 x_n + rTx_n + sx_n \to 0$. By arguments as in the proof of the complex case we find $a \in K$ such that $T^2 a + rTa + sa = 0$. The orbit of the vector a belongs to the two-dimensional subspace $L(a)$, attracting some subsequence in x_n. This contradicts the x_n being sparse. Theorem 1 is completely proved.

2. Proofs of Theorems 2 and 3

Proof of Lemma 1. Notice that $\|a\| = \|Ta\| = \|T^2 a\| = \cdots$. Indeed, this sequence is nonincreasing. The vector a is recurrent, therefore this sequence cannot decrease either. Now, for each $n \in \mathbb{N}$ the vector $T^n a$ is also recurrent, such are also linear combinations of these vectors.

So, we have $\|T(x)\| = \|x\|$ for every $x \in L(a)$. Finally, the set $T(L(a))$ is dense in $L(a)$ and therefore $T(L(a)) = L(a)$. □

Proof of Theorem 2. Assume that $\|T\| \leqslant 1$, rescaling X by the equivalent norm

$$\|x\| := \sup\{\|x\|, \|Tx\|, \|T^2x\|, \dots\}. \tag{4}$$

For each $x \in B_X$, there is $a \in K$ such that a is the limit points of the orbit $O(x)$. It is clear that $T^n x \to O(a)$ and a is recurrent vector.

According to Lemma 1 and Theorem 1, the set $O(a)$ lies in an invariant finite-dimensional subspace $L(a)$.

Different vectors x_i of the unit ball in X can be attracted, generally speaking, by the orbits of different vectors $a_i \in K$. It remains to prove that the orbits of all recurrent vectors lie in one and the same finite-dimensional space $L = \bigoplus L(a_i)$.

Let $a_1, \dots, a_n$ be recurrent vectors. Their orbits are relatively compact, since they are bounded and lie in finite-dimensional subspaces $L(a_i)$. One can find a sequence $n_k \to \infty$ such that $T^{n_k} a_i$ converges for each $i = 1, \dots, n$. Then for the sequence $m_k \to \infty$ of the form $n_{k+l} - n_k$ we have $T^{m_k}(a_i) \to a_i$. Thus if $a = \lambda_1 a_1 + \dots + \lambda_n a_n$, then $T^{m_k} a \to a$ and a is a recurrent vector.

So, the linear span L of the set of recurrent vectors itself consists of recurrent vectors. According to Lemma 1, $T : L \to L$ is isometry. According to Theorem 1, $\dim L < \infty$. So, for each $x \in X$ there exists $a \in L$ such that $T^n x \to O(a) \subseteq L$.

For every $x \in X$ the continuous function $\rho_x : L \to \mathbb{R}$ defined by the formula $\rho_x(a) = \liminf_n \|T^n x - T^n a\|$ attains its minimum 0 at a unique point $a(x) \in L$. Clearly, $\|T^n x - T^n a(x)\| \to 0$. Linearity and boundedness $A : x \mapsto a(x)$ are obvious. Put $X_0 = \ker A \subseteq X$, i.e. $x \in X_0 \Leftrightarrow T^n x \to 0$. The decomposition $X = X_0 \oplus L$ corresponds to the condition (2). Theorem 2 is proved. □

Proof of Theorem 3. The set $\tilde{K} = \bigcup_{t \in [0,1]} T_t(K) \subseteq X$ is compact, since it is the image of the compact set $K \times [0, 1]$ under the map $f(x, t) = T_t x$. It is easy to see that $\tilde{K}$ is occasionally attracting for semigroup of powers $\{T_1, T_2, \dots\}$, i.e. the operator $T_1 : X \to X$ satisfies the conditions of Theorem 2. Let L be a finite-dimensional subspace attracting X under the action of integer powers of T_1, i.e. $T_n x \to L$ for each $x \in X$. Show that $T_t x \to L$ for every x.

Suppose the contrary. Then there exist a number $\varepsilon > 0$ and a sequence $t_n \in \mathbb{R}$, $t_n \to \infty$, such that $\rho(T_{t_n} x, L) > \varepsilon$. Denote by $[t_n]$ and $\{t_n\}$ the integer and fractional parts of the number t_n. It is possible to assume that $\{t_n\} \to \beta \in [0, 1]$ by switching to a subsequence. Then

$$T_{t_n} x = T_{[t_n]+\{t_n\}} x = T_{[t_n]} T_{\{t_n\}} x \sim T_{[t_n]} T_\beta x \to L.$$

A contradiction. Theorem 3 is proved. □

3. Application to supercyclic operators

Proof of Lemma 4. If $T^n a \nrightarrow 0$, then there exist a *bounded* sequence of scalars c_k and a sequence of powers $l_k \to \infty$ such that $c_k T^{l_k} a \to a$. Choose a subsequence m_k such that $c_k \to c$ and $c T^{m_k} a \to a$. Clearly, $|c| = 1$. In this case, $c^2 T^{2m_k} a \to a$, $c^3 T^{3m_k} \to a, \dots$. But 1 is a limit point of the set $\{c^m \mid m \in \mathbb{N}\}$, therefore a is a limit point of the set $\{T^{m \cdot n_k} a \mid m, k \in \mathbb{N}\}$. □

Proof of Theorem 4. Rescaling X by the norm (4), we may suppose that $\|T\| \leqslant 1$. Let a be supercyclic. In particular, a is cyclic, i.e. the span$(O(a))$ is dense in X.

Assume that $T^n a \nrightarrow 0$. By Lemma 4, a is a recurrent vector, therefore $T : X \to X$ is an isometry by Lemma 1.

For any $x \in B_X$ there exist λ_k, $|\lambda_k| \leqslant 1$, and $n_k \to \infty$ such that $\|\lambda_k T^{n_k} y - x\| \to 0$ or, equivalently, $\|\lambda_k y - T^{-n_k} x\| \to 0$, therefore the set $K = \{\lambda y \mid |\lambda| \leqslant 1\}$ is an occasionally compact set for the *isometry* T^{-1}. A contradiction with Theorem 1.

Thus $T^n a \to 0$. But in this case $T^n x \to 0$ for every x. Indeed, for each $\varepsilon > 0$, there is a vector of the form $cT^k(a)$ that is ε-closed to x. Iterating T, we infer $cT^{k+n}(a) \to_{n\to\infty} 0$; consequently, $\|T^n x\| < \varepsilon$ for large n. Hence, $T^n x \to 0$. $\square$

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